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Combinatorial Characterization of Upward Planarity

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Abstract
We give a combinatorial characterization of upward planar graphs in terms of upward planar orders, which are special linear extensions of edge posets.

Keywords Upward planar graph · Planar st graph · Upward planar order

Mathematics Subject Classification 05C10 · 06A99

1 Introduction

A planar drawing of a directed graph is upward if all edges increase monotonically in the vertical direction (or other fixed direction). A directed graph is called upward planar if it admits an upward planar drawing; see Fig. 1 for example. Clearly, an upward planar graph is necessarily acyclic. A directed graph together with an upward planar drawing is called an upward plane graph. They are commonly used to represent hierarchical structures, such as PERT networks, Hasse diagrams, family trees, etc., and have been extensively studied in the fields of graph theory, graph drawing algorithms, and ordered set theory (see, e.g., [5] for a review).

A first simple characterization of upward planarity was given independently by Di Battista and Tamassia [3] and Kelly [8]. They characterized upward planar graphs as spanning subgraphs of planar st graphs, as shown in Fig. 2, where a planar st graph is a directed planar graph with exactly one source s, exactly one sink t, and a distinguished edge e connecting s, t.

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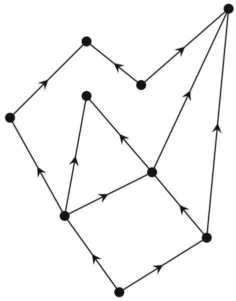


Fig. 1 An upward planar graph

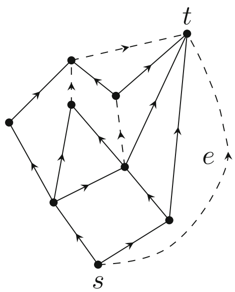


Fig. 2 A planar st graph with the graph in Fig. 1 as spanning subgraph

Another fundamental characterization of upward planar graphs was given in [1,2] by means of bimodal planar drawings [5] and consistent assignments of sources and sinks to faces. Aware of these elegant and important combinatorial characterizations, in this paper, we will study upward planarity in a different approach.

The notion of a processive plane graph (called a *PPG* for short, see Definition 2.1) was introduced in [6] as a graphical tool for tensor calculus in semi-groupal categories. It turns out that a PPG is essentially equivalent to an upward plane st graph, as shown in Fig. 3.

The notion of a progressive plane graph, with that of a PPG as a special case, was introduced by Joyal and Street in [7] as a graphical tool for tensor calculus in monoidal categories. Similar to the above characterization of Di Battista and Tamassia [3] and Kelly [8], any progressive plane graph can be extended (in a non-unique way) to a PPG, as shown in Fig. 4. It is clear that a progressive plane graph is essentially an upward planar graph (possibly with isolated vertices).

One main result in [6] is that a PPG can be characterized in terms of the notion of a planar order, which is a special linear extension of the edge poset (short for partially ordered set) of the underlying directed graph (Theorem 2.5). Based on and to generalize this result, we will give a similar characterization of upward planar graphs. Precisely, we introduce the notion of an upward planar order (Definition 3.1) for an acyclic directed graph G, which is a generalization of that of a planar order, and show that G is upward planar if and only if it has an upward planar order (Theorem 6.1). We summarize the conceptual framework as follows.

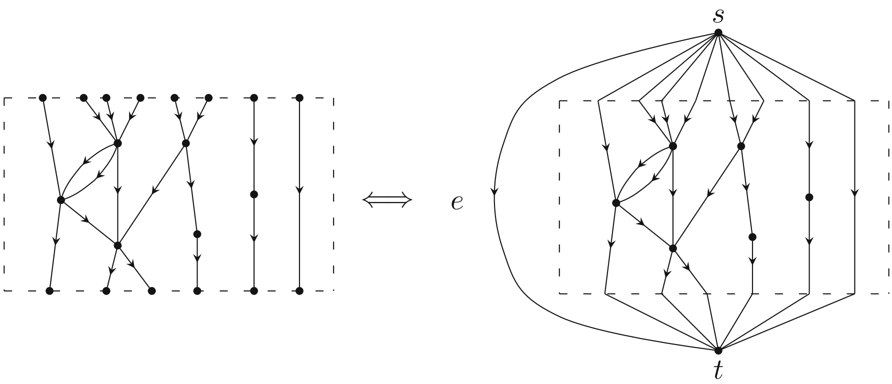


Fig. 3 A PPG and its associated upward plane st graph

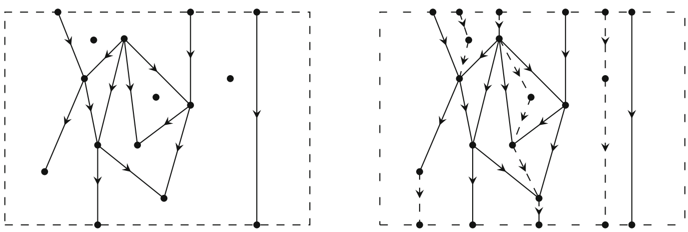


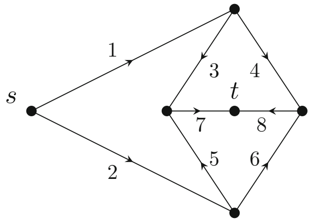
Fig. 4 A progressive plane graph and one of its PPG-extentions

Category theory	Graph theory	Combinatorial characterization
PPG	Upward plane st graph	Planar order
Progressive plane graph	Upward plane graph	Upward planar order

To prove our main result, Theorem 6.1, we put in much effort to analyze the relationship between the notion of a planar order and that of an upward planar order (Theorem 4.6) and especially introduce the notion of a canonical processive planar extension (*CPP* extension for short, Definition 5.1), which is the crucial link between our combinatorial characterization of PPGs and that of upward plane graphs. Our strategy of justifying our characterization of upward planar graphs by means of CPP extensions and combinatorial characterization of processive plane graphs is a combinatorial formulation of PPG-extensions of progressive plane graphs and in a sense, a refinement of the work of Di Battista and Tamassia [3] and Kelly [8].

One advantage of our approach to study upward planarity is that it admits a composition theory of upward planar orders just as that of planar orders in [6], which provides a practical way to compute an associated upward planar order of an upward plane graph. The composition theory will be presented elsewhere. Another advantage is that our characterization sheds some light on the long-standing problem [9,10] of finding a topological theory of posets (parallel to topological graph theory) which

Fig. 5 A directed graph which is not upward planar but with linear genus zero



should generalize upward planarity to higher genus surfaces. Observe that any linear extension of the edge poset of a connected acyclic directed graph G naturally induces a rotation system on G, or a cellular embedding of G on a surface, whose genus is called the linear genus of the linear extension. The linear genus of G is defined as the minimal linear genus of linear extensions of the edge poset of G. Our characterization means that G is upward planar implying that the linear genus of G is zero. However, the converse is not true; see Fig. 5.

It would be interesting to find possible relations between the aforementioned theory of linear genus and the higher genus theory of upward planarity proposed in [6], which is based on the graphical calculus for symmetric monoidal categories (Chapter 2 of [7]). Yet another interesting fact is that the axioms in the definition of an upward planar order can be restated as axioms for hypergraphs, or in other words, the notion of an upward planar order makes sense for hypergraphs. These directions are worth studying in the future.

The paper is organized as follows. In Sect. 2, we review the combinatorial characterization of PPGs in [6]. In Sect. 3, we introduce the notions of an upward planar order and a UPO-graph and study their basic properties. In Sect. 4, we give several new characterizations of PPGs. In Sect. 5, we introduce the notion of a CPP extension for an acyclic directed graph G. We show that there is a natural bijection between upward planar orders on G and CPP extensions of G. In Sect. 6, we justify the notion of a UPO-graph by showing that a UPO-graph has a unique upward planar drawing up to planar isotopy, and conversely, there is at least one upward planar order for any upward plane graph. We point out that our characterization of upward planarity can also be applied to characterize (non-directed) planar graphs.

2 PPG and POP-Graph

In this section, we recall the notion of a PPG and its combinatorial characterization.

Definition 2.1 A processive plane graph, or PPG, is an acyclic directed graph drawn in a plane box such that (1) all edges monotonically decrease in the vertical direction; (2) all sources and sinks are of degree one; and (3) all sources and sinks are drawn on the horizontal boundaries of the plane box.

The left of Fig. 3 shows an example. The following notion characterizes the underlying graph of a PPG.

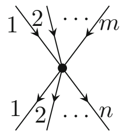


Fig. 6 Polarization of a vertex

Definition 2.2 A processive graph is an acyclic directed graph with all sources and sinks being of degree one.

Clearly, this notion is essentially equivalent to that of a PERT-graph [11] which is a directed graph with exactly one source s and exactly one sink t (the underlying graph in Fig. 5 is an example), and also equivalent to that of an st graph which is a PERT-graph with a distinguished edge e connecting s and t (the underlying directed graph in the right of Fig. 3 is an example).

A vertex is called processive if it is neither a source nor a sink. An edge of a processive graph is called an input edge if it starts from a source and output edge if it ends with a sink. We denote the set of input edges of a processive graph G by $I(G)$ and the set of output edges by $O(G)$.

A planar drawing of a processive graph G is boxed if it is drawn in a plane box with all sources of G on one of the horizontal boundaries of the plane box and all sinks of G on the other one. From the left of Fig. 3, it is easy to see that a PPG is a boxed and upward planar drawing of a processive graph. Two PPGs are equivalent if they are connected by a planar isotopy such that each intermediate planar drawing is boxed (but not necessarily upward).

For a vertex v of a directed graph G, a polarization [7] of v consists of two linear orders, one on the set $I_G(v)$ (or *I(v)* for simplicity) of incoming edges of v and the other on the set $O_G(v)$ (or *O(v)* for simplicity) of outgoing edges of v (possibly one of them is empty). A directed graph is called polarized if each vertex is equipped with a polarization. In the way shown in Fig. 6, PPGs and general upward plane graphs are polarized.

The following is a key notion in [6].

Definition 2.3 A planar order on a processive graph G is a linear order $\prec$ on the edge set $E(G)$, such that

(P_1) $e_1 \to e_2$ implies that $e_1 \prec e_2$;
(P_2) if $e_1 \prec e_2 \prec e_3$ and $e_1 \to e_3$, then either $e_1 \to e_2$ or $e_2 \to e_3$,

where $e_1 \to e_2$ denotes that there is a directed path starting from e_1 and ending with e_2.

Figure 7 shows a simple example motivating the definition. By the linearity of $\prec$, it is easy to see that (P_2) is equivalent to $(\widetilde{P_2})$: If $e_1 \prec e_2 \prec e_3$ and $t(e_1) = s(e_3)$, then either $e_1 \to e_2$ or $e_2 \to e_3$, where $s(e)$, $t(e)$ denote the starting and ending vertex of edge e, respectively.

Definition 2.4 A processive graph together with a planar order is called a planarly ordered processive graph or *POP*-graph for short.

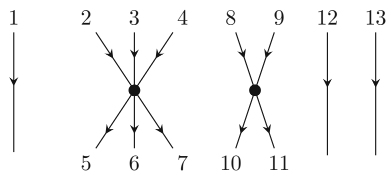


Fig. 7 A planar order

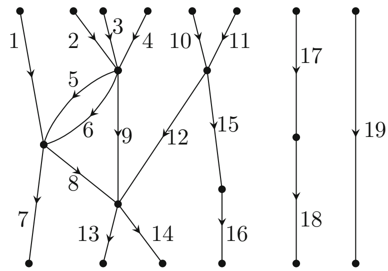


Fig. 8 A POP-graph

The following is a key result in [6].

Theorem 2.5 *There is a bijection between POP-graphs and equivalence classes of PPGs.*

Figure 8 shows the corresponding POP-graph of the PPG in the left of Fig. 3.

3 UPO–Graph

In this section, we introduce the key notion in this paper, that is, the notion of a UPO-graph and show some of its basic properties.

We first introduce some notations. Let S be a finite set with a linear order $<$. Given a subset $X \subseteq S$, we write $X^- = \min X$ and $X^+ = \max X$. The convex hull of X in S is $\overline{X} = \{y \in S | X^- \leq y \leq X^+\}$.

Definition 3.1 An upward planar order on a directed graph G is a linear order $\prec$ on $E(G)$, such that

(U_1) $e_1 \to e_2$ implies that $e_1 \prec e_2$;
(U_2) for any vertex v, $\overline{I(v)} \cap \overline{O(v)} = \emptyset$ and $\overline{E(v)} = \overline{I(v)} \sqcup \overline{O(v)}$;
(U_3) for any two vertices v_1 and v_2, $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$ implies that $\overline{I(v_1)} \subseteq \overline{I(v_2)}$, and $O(v_1) \cap \overline{O(v_2)} \neq \emptyset$ implies that $\overline{O(v_1)} \subseteq \overline{O(v_2)}$.

Figure 9 is a typical example motivating this definition.

Definition 3.2 A directed graph together with an upward planar order is called an upward planarly ordered graph or *UPO*–graph for short.

Any UPO–graph must be acyclic. Obviously, $(U_1) = (P_1)$, say $\prec$ is a linear extension of $\to$. (U_2), under (U_1), is equivalent to $O(v)^- = I(v)^+ + 1$ (with respect to $\prec$) for any processive vertex v.

The following lemma is an easy consequence of (U_3).

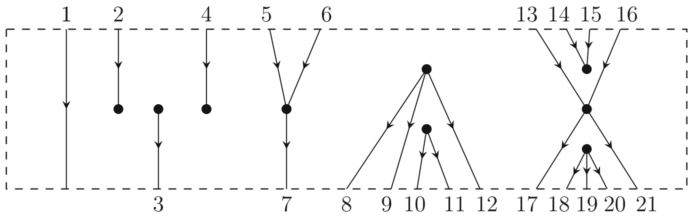


Fig. 9 An upward planar order

Lemma 3.3 *Let G be an acyclic directed graph, $\prec$ a linear order on $E(G)$, and v_1 and v_2 be two vertices of G. If $\prec$ satisfies (U_3), then*

(1) $\overline{I(v_1)} \cap \overline{I(v_2)} \neq \emptyset$ *implies that either* $\overline{I(v_1)} \subseteq \overline{I(v_2)}$ *or* $\overline{I(v_2)} \subseteq \overline{I(v_1)}$.
(2) $\overline{O(v_1)} \cap \overline{O(v_2)} \neq \emptyset$ *implies that either* $\overline{O(v_1)} \subseteq \overline{O(v_2)}$ *or* $\overline{O(v_2)} \subseteq \overline{O(v_1)}$.

Proof We only prove (1). The proof of (2) is similar. Both $\overline{I(v_1)}$ and $\overline{I(v_2)}$ are intervals of $(E(G), \prec)$, then $\overline{I(v_1)} \cap \overline{I(v_2)} \neq \emptyset$ implies that either $I(v_1)^+ \in \overline{I(v_2)}$ or $I(v_2)^+ \in \overline{I(v_1)}$. In the former case, notice that $I(v_1)^+ \in I(v_1)$, so $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$. By (U_3), $\overline{I(v_1)} \subset \overline{I(v_2)}$. Similarly, in the latter case, $\overline{I(v_2)} \subset \overline{I(v_1)}$. □

An embedding of directed graphs $\phi\colon G_1 \to G_2$ consists of a pair of injections $\phi_0\colon V(G_1) \to V(G_2)$ and $\phi_1\colon E(G_1) \to E(G_2)$, such that $s(\phi_1(e)) = \phi_0(s(e))$ and $t(\phi_1(e)) = \phi_0(t(e))$ for any $e \in E(G_1)$. In this case, G_1 is called a subgraph of G_2. We freely identify the vertices and edges of G_1 with their images and view $V(G_1) \subseteq V(G_2)$, $E(G_1) \subseteq E(G_2)$.

The following proposition shows that (U_1) and (U_3) are hereditary.

Proposition 3.4 *Let G be an acyclic directed graph with a linear order $\prec$ on $E(G)$, H a subgraph of G and $\prec_H$ the linear order on $E(H)$ induced from $\prec$. Then,*

(1) $\prec$ *satisfies* (U_1) *implying that* $\prec_H$ *satisfies* (U_1).
(2) $\prec$ *satisfies* (U_3) *implying that* $\prec_H$ *satisfies* (U_3).

Proof (1) is a direct consequence of the facts that for $e_1, e_2 \in E(H)$, $e_1 \to e_2$ in H if and only if $e_1 \to e_2$ in G, and that $e_1 \prec_H e_2$ if and only if $e_1 \prec e_2$.

Now we prove (2) by contradiction. Suppose there exist $v_1, v_2 \in V(H)$, such that $I_H(v_1) \cap \overline{I_H(v_2)} \neq \emptyset$ and $\overline{I_H(v_1)} \nsubseteq \overline{I_H(v_2)}$. On the one hand, $I_H(v_1) \cap \overline{I_H(v_2)} \neq \emptyset$ implies that $I_G(v_1) \cap \overline{I_G(v_2)} \neq \emptyset$, then by (U_3) of $\prec$, $\overline{I_G(v_1)} \subseteq \overline{I_G(v_2)}$.

On the other hand, we must have $\overline{I_G(v_2)} \subseteq \overline{I_G(v_1)}$. In fact, $I_H(v_1) \cap \overline{I_H(v_2)} \neq \emptyset$ means that there is an edge $e \in I_H(v_1)$ such that $I_H(v_2)^- \preceq_H e \preceq_H I_H(v_2)^+$, and $\overline{I_H(v_1)} \nsubseteq \overline{I_H(v_2)}$ means that there is an edge $h \in \overline{I_H(v_1)}$ such that $h \prec_H I_H(v_2)^-$ or $I_H(v_2)^+ \prec_H h$. In the former case, $I_H(v_2)^- \in [h, e] \subseteq \overline{I_H(v_1)}$. In the latter case, $I_H(v_2)^+ \in [e, h] \subseteq \overline{I_H(v_1)}$. So in both cases, $I_H(v_2) \cap \overline{I_H(v_1)} \neq \emptyset$, and hence $I_G(v_2) \cap \overline{I_G(v_1)} \neq \emptyset$. Then, by (U_3) of $\prec$, $\overline{I_G(v_2)} \subseteq \overline{I_G(v_1)}$.

In summary, $\overline{I_G(v_1)} = \overline{I_G(v_2)}$, that is, $v_1 = v_2$, which contradicts the assumption $\overline{I_H(v_1)} \nsubseteq \overline{I_H(v_2)}$. Similarly, we can show that $O_H(v_1) \cap \overline{O_H(v_2)} \neq \emptyset$ implies $\overline{O_H(v_1)} \subseteq \overline{O_H(v_2)}$. □

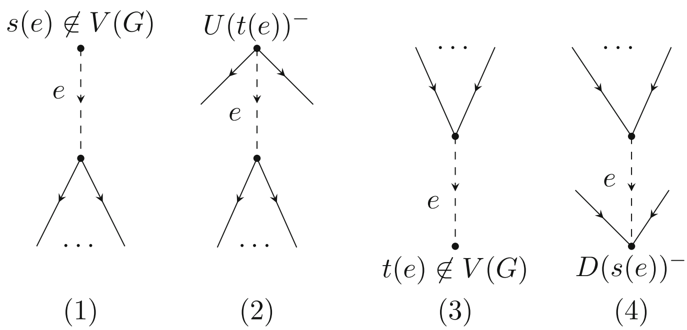


Fig. 10 Local configurations of e in Theorem 3.5

Let $(G, \prec)$ be a UPO–graph and v a vertex of G. We set

$$U(v) = \begin{cases} \{w | w \in V(G),\ \overline{O(v)} \subsetneq \overline{O(w)}\}, & \text{if } O(v) \neq \emptyset; \\ \emptyset, & \text{otherwise;} \end{cases}$$

$$D(v) = \begin{cases} \{w | w \in V(G),\ \overline{I(v)} \subsetneq \overline{I(w)}\}, & \text{if } I(v) \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

We define an order $<$ on $U(v)$ as follows. For any $w_1, w_2 \in U(v)$, $w_1 < w_2$ if $\overline{O(w_1)} \subsetneq \overline{O(w_2)}$. By Lemma 3.3, $<$ is a linear order on $U(v)$. Similarly, $D(v)$ is a linearly ordered set under the order that $w_1 < w_2$ in $D(v)$ if $\overline{I(w_1)} \subsetneq \overline{I(w_2)}$.

The following theorem shows that when suitable vertices and edges are added to a UPO–graph, the resulting graph will admit a unique extended upward planar order.

Theorem 3.5 *Let $(G, \prec_G)$ be a UPO–graph. $S(G)$ and $T(G)$ are the sets of sources and sinks of G, respectively. Assume that Γ is a directed graph obtained by adding a new edge e to G in any one of the following ways (see Fig. 10):*

(1) $t(e) \in S(G)$, $U(t(e)) = \emptyset$ *in* $(G, \prec_G)$, *and* $s(e) \notin V(G)$;
(2) $t(e) \in S(G)$, $U(t(e)) \neq \emptyset$ *in* $(G, \prec_G)$, *and* $s(e) = U(t(e))^-$;
(3) $s(e) \in T(G)$, $D(s(e)) = \emptyset$ *in* $(G, \prec_G)$, *and* $t(e) \notin V(G)$;
(4) $s(e) \in T(G)$, $D(s(e)) \neq \emptyset$ *in* $(G, \prec_G)$, *and* $t(e) = D(s(e))^-$.

Then, there exists a unique upward planar order $\prec_\Gamma$ on Γ, whose restriction is $\prec_G$.

Proof The uniqueness follows from (U_2) of $\prec_\Gamma$. In fact, in cases (1) and (2), $e = O(t(e))^- - 1$, and in cases (3) and (4), $e = I(s(e))^+ + 1$. In this way, the linear order $\prec_\Gamma$ on $E(\Gamma) = E(G) \sqcup \{e\}$ is uniquely defined. To show the existence, it suffices to show that $\prec_\Gamma$ is an upward planar order.

(i) First, we check (U_1) for $\prec_\Gamma$. In case (1), $e \in I(\Gamma)$, so we only need to show that $e \to e_1$ implies that $e \prec_\Gamma e_1$. In fact, $e \to e_1$, implying that there is an edge $e_2 \in O(t(e))$ such that $e_2 \to e_1$ in G, so $e_2 \prec_G e_1$ and hence $e_2 \prec_\Gamma e_1$. Then, $e = O(t(e))^- - 1 \prec_\Gamma e_2 \prec_\Gamma e_1$.

In case (2), we only need to show that $e_1 \to e$ and $e \to e_2$ implying that $e_1 \prec_\Gamma e_2$. Set $w = U(t(e))^-$. On the one hand, $\overline{O(t(e))} \subsetneq \overline{O(w)}$ and $e = O(t(e))^- - 1$ implying that $O(w)^- \prec_\Gamma e$. By $e_1 \to e$, $e_1 \to O(w)^-$ in G, so $e_1 \prec_G O(w)^-$ and hence $e_1 \prec_\Gamma O(w)^- \prec_\Gamma e$. On the other hand, $\overline{O(t(e))} \subsetneq \overline{O(w)}$ implying the nonexistence of a direct path in G that starts from $t(e)$ and ends with w, so just as case (1), $e \to e_2$ implies that there is an edge $e_3 \in O(t(e))$ such that $e_3 \to e_2$ in G, which implies that $e \prec_\Gamma e_3 \prec_\Gamma e_2$. In summary, $e_1 \prec_\Gamma O(w)^- \prec_\Gamma e \prec_\Gamma e_3 \prec_\Gamma e_2$. The proofs in cases (3) and (4) are similar.

(ii) Under (U_1), (U_2) is equivalent to $O(v)^- = I(v)^+ + 1$ which is obvious from the construction of $\prec_\Gamma$.

(iii) Now we are left to check (U_3). If $v_1 = v_2$ or $v_1 \in V(\Gamma) - V(G)$ or $v_2 \in V(\Gamma) - V(G)$, (U_3) is trivial. So we assume v_1 and v_2 are different vertices of G such that $I_\Gamma(v_1) \cap \overline{I_\Gamma(v_2)} \neq \emptyset$. There are two possibilities for v_1.

If $e \notin I_\Gamma(v_1)$, then $I_\Gamma(v_1) = I_G(v_1)$. So $I_G(v_1) \cap \overline{I_G(v_2)} = I_G(v_1) \cap (\overline{I_G(v_2)} \sqcup \{e\}) \supseteq I_\Gamma(v_1) \cap \overline{I_\Gamma(v_2)} \neq \emptyset$. Applying (U_3) for $\prec_G$, we have $\overline{I_G(v_1)} \subseteq \overline{I_G(v_2)}$. Then, $I_\Gamma(v_1) = I_G(v_1) \subseteq \overline{I_G(v_1)} \subseteq \overline{I_G(v_2)} \subseteq \overline{I_\Gamma(v_2)}$, which implies that $\overline{I_\Gamma(v_1)} \subseteq \overline{I_\Gamma(v_2)}$.

Otherwise, $e \in I_\Gamma(v_1)$, there are three cases. In cases (1) and (2), $I_\Gamma(v_1) = \overline{I_\Gamma(v_1)} = \{e\}$ and hence $\overline{I_\Gamma(v_1)} \subseteq \overline{I_\Gamma(v_2)}$. In case (4), $v_1 = t(e) = D(s(e))^-$. Note that $I_\Gamma(v_1) = I_G(v_1) \sqcup \{e\}$, so $I_\Gamma(v_1) \cap \overline{I_\Gamma(v_2)} \neq \emptyset$ implies that $I_G(v_1) \cap \overline{I_\Gamma(v_2)} \neq \emptyset$ or $e \in \overline{I_\Gamma(v_2)}$. In the former case, since $e \notin I_G(v_1)$, so $I_G(v_1) \cap \overline{I_G(v_2)} \neq \emptyset$, which, by applying (U_3) for $\prec_G$, implies that $\overline{I_G(v_1)} \subseteq \overline{I_G(v_2)}$, and hence $\overline{I_\Gamma(v_1)} \subseteq \overline{I_\Gamma(v_2)}$. In the later case, note that $e = I_G(s(e))^+ + 1$ and $e \notin I_\Gamma(v_2)$ (by $v_1 \neq v_2$), so $e \in \overline{I_\Gamma(v_2)}$ implies that $I_G(s(e))^+ \in \overline{I_\Gamma(v_2)}$. So $I_G(s(e)) \cap \overline{I_G(v_2)} \neq \emptyset$, which, by applying (U_3) for $\prec_G$, implies that $\overline{I_G(s(e))} \subseteq \overline{I_G(v_2)}$, that is, $v_2 \in D(s(e))$. Since $v_1 = D(s(e))^-$, so $\overline{I_G(v_1)} \subseteq \overline{I_G(v_2)}$, which implies that $\overline{I_\Gamma(v_1)} \subseteq \overline{I_\Gamma(v_2)}$.

Dually, we can show that for any different $v_1, v_2 \in V(G)$, $\overline{O_\Gamma(v_1)} \subseteq \overline{O_\Gamma(v_2)}$ provided that $O_\Gamma(v_1) \cap \overline{O_\Gamma(v_2)} \neq \emptyset$. The proof is completed. □

4 Characterizations of POP-Graphs

In this section, we introduce some constraints for UPO-graphs and list some lemmas which are useful for proving new characterizations of POP-graphs.

Lemma 4.1 *Let G be a processive graph with a linear order $\prec$ on $E(G)$. Then,*

(1) *for any vertices $v_1 \neq v_2$ of G, $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$ implies that v_2 is processive.*
(2) *for any vertices $v_1 \neq v_2$ of G, $O(v_1) \cap \overline{O(v_2)} \neq \emptyset$ implies that v_2 is processive.*

Proof We only prove (1). By assumption $v_1 \neq v_2$ and $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$, we know that the degree of v_2 is not equal to one. Since G is a processive graph, v_2 must be processive. The proof of (2) is similar. □

Definition 4.2 A UPO-graph is called anchored if (A) for any different vertices v_1, v_2 of G, $\emptyset \neq \overline{I(v_1)} \subset \overline{I(v_2)}$ implies that $v_1 \to v_2$; $\emptyset \neq \overline{O(v_1)} \subset \overline{O(v_2)}$ implies that $v_2 \to v_1$, where $v \to w$ denotes that there is a directed path starting from v and ending with w.

In Sect. 6, we will show that any UPO-graph has an upward planar drawing, which is called the geometric realization of the UPO-graph. In the geometric realization of an anchored UPO-graph, all sources and sinks are drawn on the boundary of the external face; see Remark 6.4 for explanation.

Lemma 4.3 *Let $(G, \prec)$ be a POP-graph. Then, $\prec$ satisfies (A).*

Proof We only prove the first part of (A). The proof of the second part is similar and we omit it here. Let $v_1 \neq v_2$ be two vertices of G, such that $\overline{I(v_1)} \subset \overline{I(v_2)}$ and $\overline{I(v_1)} \neq \emptyset$. Then, $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$, and therefore by Lemma 4.1 (1), v_2 is processive.

Take $e \in I(v_1) \subset \overline{I(v_2)}$. Then, $I(v_2)^- \prec e \prec I(v_2)^+$, and hence $I(v_2)^- \prec e \prec O(v_2)^-$, where the last equality follows from (P_1) and the fact $I(v_2)^+ \to O(v_2)^-$. By (P_2), $I(v_2)^- \to O(v_2)^-$ implies that either $I(v_2)^- \to e$ or $e \to O(v_2)^-$. If $I(v_2)^- \to e$, then $I(v_2)^+ \to e$, and by (P_1), $I(v_2)^+ \prec e$, a contradiction. Thus, we must have $e \to O(v_2)^-$, which implies that $v_1 \to v_2$. □

Lemma 4.4 *Let $(G, \prec)$ be a UPO–graph. If G is a processive graph, then (A) is equivalent to the following condition:*

(U_4) for any processive vertex v of G, $I(G) \cap \overline{O(v)} = \emptyset$ and $O(G) \cap \overline{I(v)} = \emptyset$.

Proof $(A) \Longrightarrow (U_4)$. We prove this by contradiction. Suppose there is a processive vertex v of G such that $I(G) \cap \overline{O(v)} \neq \emptyset$. Take $i \in I(G) \cap \overline{O(v)}$ and set $w = s(i)$. Clearly, by definition, $\overline{O(w)} = \{i\}$. Then, $\overline{O(w)} \subset \overline{O(v)}$ and $\overline{O(w)} \neq \emptyset$. Thus, by (A) we have $v \to w$, which contradicts $i \in I(G)$. Similarly, we can prove that for any processive vertex v of G, $O(G) \cap \overline{I(v)} = \emptyset$.

$(U_4) \Longrightarrow (A)$. Let v_1, v_2 be two different vertices of G with $\overline{I(v_1)} \subset \overline{I(v_2)}$ and $\overline{I(v_1)} \neq \emptyset$, we want to prove that $v_1 \to v_2$. First, $\overline{I(v_1)} \subset \overline{I(v_2)}$ and $\overline{I(v_1)} \neq \emptyset$ imply that $I(v_1) \cap \overline{I(v_2)} = I(v_1) \neq \emptyset$, and hence by Lemma 4.1 (1), v_2 is a processive vertex.

Next, we claim that v_1 must not be a sink. If not, by the fact that G is a processive graph, $\overline{I(v_1)} = \{e\} \subseteq I(G)$, which implies that $I(G) \cap \overline{I(v_2)} \supseteq \{e\} \neq \emptyset$, contradicting (U_4). So v_1 is a processive vertex.

Now by (U_2), $\overline{I(v_1)} \subset \overline{I(v_2)}$ implies that $I(v_2)^- \prec O(v_1)^- = I(v_1)^+ + 1 \preceq I(v_2)^+$. If $O(v_1)^- = I(v_2)^+$, then $v_1 \to v_2$ and we complete the proof. Otherwise, $I(v_2)^- \prec O(v_1)^- \prec I(v_2)^+$, which implies that $I(w_1) \cap \overline{I(v_2)} \neq \emptyset$, where $w_1 = t(O(v_1)^-)$. Clearly, $v_1 \to w_1$ and by (U_3), $\overline{I(w_1)} \subset \overline{I(v_2)}$. If $w_1 \to v_2$, then $v_1 \to v_2$ and we complete the proof. Otherwise, note that $O(v_1)^- \in \overline{I(w_1)} \neq \emptyset$, similar as v_1, w_1 must not be a sink, so we can repeat the above procedure to find $w_2 \to w_3 \to \cdots$, until we find a w_k $(k \geq 1)$ such that $w_k \to v_2$. Since G has only finite vertices and the above procedure never reaches a sink, so such a w_k must exist, and hence we have $v_1 \to v_2$.

Similarly, we can prove that $\overline{O(v_1)} \subset \overline{O(v_2)}$ and $\overline{O(v_1)} \neq \emptyset$ imply $v_2 \to v_1$. □

Lemma 4.5 *Let G be a processive graph and $\prec$ a linear order on $E(G)$. If $\prec$ satisfies (U_1) and (U_2), then the following conditions are equivalent:*

$(\widetilde{P}_2)$ if $e_1 \prec e_2 \prec e_3$ and $t(e_1) = s(e_3)$, then either $e_1 \to e_2$ or $e_2 \to e_3$.

(P_3) for any two vertices v_1 and v_2, $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$ implies that $v_1 \to v_2$ and $O(v_1) \cap \overline{O(v_2)} \neq \emptyset$ implies that $v_2 \to v_1$.

Proof $(\widetilde{P}_2) \Longrightarrow (P_3)$. We only prove the first part of (P_3), the proof of the second part is similar. First by Lemma 4.1 (1), $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$ implies that v_2 is processive.

Now take $e \in I(v_1) \cap \overline{I(v_2)}$, then $I(v_2)^- \prec e \prec I(v_2)^+ \prec O(v_2)^-$, where the last equality follows from (U_2). Clearly, $t(I(v_2)^-) = v_2 = s(O(v_2)^-)$, then by $(\widetilde{P}_2)$ we have either $I(v_2)^- \to e$ or $e \to O(v_2)^-$. If $I(v_2)^- \to e$, then $I(v_2)^+ \to e$, which contradicts $e \prec I(v_2)^+$ and (U_1). So we must have $e \to O(v_2)^-$, which implies that $v_1 \to v_2$.

$(P_3) \Longrightarrow (\widetilde{P}_2)$. Assume $e_1 \prec e_2 \prec e_3$ and $v = t(e_1) = s(e_3)$. By (U_2), $\overline{E(v)} = \overline{I(v)} \sqcup \overline{O(v)}$. Then, $e_2 \in [e_1, e_3] \subseteq \overline{E(v)}$ implies that either $e_2 \in \overline{I(v)}$ or $e_2 \in \overline{O(v)}$. If $e_2 \in \overline{I(v)}$, then $e_2 \in I(t(e_2)) \cap \overline{I(v)} \neq \emptyset$. So by (P_3), we have $t(e_2) \to v$, hence $e_2 \to e_3$. Similarly, $e_2 \in \overline{O(v)}$ implies that $e_1 \to e_2$. □

Now we give several characterizations of POP-graphs.

Theorem 4.6 *Let G be a processive graph with a linear order $\prec$ on $E(G)$ satisfying (P_1). Then, the following statements are equivalent*:

(1) $(G, \prec)$ *is a POP-graph.*
(2) $\prec$ *satisfies* (U_2) *and* (P_3).
(3) $(G, \prec)$ *is an anchored UPO–graph.*
(4) $\prec$ *satisfies* (U_2), (U_3) *and* (U_4).

Proof (1) $\Longleftrightarrow$ (2). By Lemma 4.5 and the fact that $(P_2) \Longleftrightarrow (\widetilde{P}_2)$, we see that $(P_2) \Longleftrightarrow (P_3)$ under (U_1) and (U_2). Since $\prec$ satisfies $(P_1) = (U_1)$, then to prove (1) $\Longleftrightarrow$ (2) we only need to prove $(P_1) + (P_2) \Longrightarrow (U_2)$.

In fact, let v be a processive vertex of G, $e_1 = I(v)^+$ and $e_2 = O(v)^-$. Clearly, $e_1 \to e_2$, and by (P_1), $e_1 \prec e_2$. Now we prove $e_2 = e_1 + 1$ by contradiction. Suppose there exists an edge e with $e_1 \prec e \prec e_2$, then by (P_2) we have either $e_1 \to e$ or $e \to e_2$. If $e_1 \to e$, then there must exist an edge $e' \in O(v)$ such that $e' \to e$ or $e' = e$, which follows $e' \preceq e$ by (P_1). Hence, $e' \prec e_2$, which contradicts the facts that $e' \in O(v)$ and $e_2 = O(v)^-$. Similarly, $e \to e_2$ also leads a contradiction.

(1) $\Longrightarrow$ (3). We have proved $(P_1) + (P_2) \Longrightarrow (U_2)$ and Lemma 4.3 shows that $(P_1)+(P_2) \Longrightarrow (A)$; thus, to prove (1) $\Longrightarrow$ (3), we only need to prove $(P_1)+(P_2) \Longrightarrow (U_3)$. By (1) $\Longleftrightarrow$ (2), it suffices to show $(P_3) \Longrightarrow (U_3)$.

We prove this by contradiction. Suppose $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$ and $\overline{I(v_1)} \nsubseteq \overline{I(v_2)}$. On the one hand, by (P_3), $I(v_1) \cap \overline{I(v_2)} \neq \emptyset$ implies $v_1 \to v_2$. On the other hand, $\overline{I(v_1)} \cap \overline{I(v_2)} \neq \emptyset$ implies that $\overline{I(v_1)} \cap \overline{I(v_2)} \neq \emptyset$. Assume $\overline{I(v_1)} = [e_1, e_2]$ and $\overline{I(v_2)} = [h_1, h_2]$. Since $\overline{I(v_1)} \nsubseteq \overline{I(v_2)}$, so we have either $e_1 \prec h_1 \prec e_2$ or $e_1 \prec h_2 \prec e_2$. Both cases imply $I(v_2) \cap \overline{I(v_1)} \neq \emptyset$; then, by (P_3), we have $v_2 \to v_1$, a contradiction with the acyclicity of G. The second part of (U_3) can be proved similarly.

(3) $\Longrightarrow$ (2). This is a direct consequence of the fact that $(U_3) + (A) \Longrightarrow (P_3)$.

(3) $\Longleftrightarrow$ (4). This is a direct consequence of Lemma 4.4. □

5 CPP Extension

In this section, we introduce the notion of a CPP extension for a directed graph and show that CPP extensions are naturally in bijective with upward planar orders.

Definition 5.1 A canonical processive planar extension, or *CPP*-extension, of a directed graph G is a POP-graph $(\Gamma, \prec)$ together with an embedding $\phi\colon G \to \Gamma$, such that

(E_1) $\phi_0(V(G)) = V(\Gamma) - (S(\Gamma) \sqcup T(\Gamma))$;
(E_2) $|E(\Gamma)| = |E(G)| + |S(G)| + |T(G)|$, where $|X|$ denotes the cardinality of X;
(E_3) $I(\phi_0(v)) \cap O(\phi_0(w)) = \emptyset$ for any $v \in S(G)$ and $w \in T(G)$;
(E_4) $e \in E(\Gamma) - (\phi_1(E(G)) \cup I(\Gamma) \cup O(\Gamma))$ implies that either $O(s(e))^- \prec e \prec O(s(e))^+$ or $I(t(e))^- \prec e \prec I(t(e))^+$.

Clearly, G must be acyclic if it has a CPP extension. Any CPP-extension of G is obtained from G by adding some new vertices and edges, with local configurations as listed in Fig. 10. Since Γ is processive, (E_1) says that the vertices of G exactly correspond to the processive vertices of Γ. (E_2) says that the number of newly added edges is exactly the number $|S(G)| + |T(G)|$ of sources and sinks, and (E_3) says that any newly added edge should not connect a source and a sink of G. (E_4) says that if a newly added edge is neither an input nor output edge of Γ, then its local configuration should be the case (2) or (4) in Fig. 10.

Remark 5.2 In general, CPP-extensions may not exist, and it may not be unique even if it exists, while for a UPO–graph, there exists a unique compatible CPP-extension; see Theorem 5.5.

The following lemma characterizes the newly added edges.

Lemma 5.3 *Let G be an acyclic directed graph and $\phi\colon G \to (\Gamma, \prec)$ a CPP-extension. Then,*

$$\bigsqcup_{v\in S(G)} I(\phi_0(v)) \bigsqcup \bigsqcup_{w\in T(G)} O(\phi_0(w)) = E(\Gamma) - \phi_1(E(G)),$$

and $|I(\phi_0(v))| = 1$ *for any* $v \in S(G)$, *and* $|O(\phi_0(w))| = 1$ *for any* $w \in T(G)$.

Proof Assume $S(G) = \{v_1, \dots, v_m\}$ and $T(G) = \{w_1, \dots, w_n\}$. By (E_1), $\phi_0(v_k)$ $(1 \le k \le m)$ and $\phi_0(w_l)$ $(1 \le l \le n)$ are all processive vertices of Γ. Thus, $I(\phi_0(v_k))$ $(1 \le k \le m)$ and $O(\phi_0(w_l))$ $(1 \le l \le n)$ are not empty. Clearly,

$$\bigcup_{1\le k\le m} I(\phi_0(v_k)) \bigcup \bigcup_{1\le l\le n} O(\phi_0(w_l)) \subseteq E(\Gamma) - \phi_1(E(G)),$$

and $I(\phi_0(v_k)) \cap I(\phi_0(v_l)) = \emptyset$ for any $1 \le k < l \le m$ and $O(\phi_0(w_k)) \cap O(\phi_0(w_l)) = \emptyset$ for any $1 \le k < l \le n$. Then , by (E_2) the cardinal number of $E(\Gamma) - \phi_1(E(G))$ is $m + n$ and by (E_3) the cardinal number of $\bigcup_{1\le k\le m} I(\phi_0(v_k)) \bigcup \bigcup_{1\le l\le n} O(\phi_0(w_l))$ is also $m + n$. So we have $\bigsqcup_{1\le k\le m} I(\phi_0(v_k)) \bigsqcup \bigsqcup_{1\le l\le n} O(\phi_0(w_l)) = E(\Gamma) - \phi_1(E(G))$ and $|I(\phi_0(v_k))| = 1\,(1 \le k \le m)$, $|O(\phi_0(w_l))| = 1\,(1 \le l \le n)$. □

Let $\prec_G$ be the linear order on $E(G)$ induced from $\prec$. The following lemma is a direct consequence of (E_4), which says that some properties of an edge of G with respect to $\prec_G$ are preserved by the embedding $\phi : G \to \Gamma$.

Lemma 5.4 *Let G be an acyclic directed graph and $\phi\colon G \to (\Gamma,\prec)$ a CPP-extension of G. For any edge e of G, we set $e' = \phi_1(e)$. Then, we have*:

(1) $e = I(t(e))^-$ *in* $(G, \prec_G) \iff e' = I(t(e'))^-$ *in* $(\Gamma, \prec)$;
(2) $e = I(t(e))^+$ *in* $(G, \prec_G) \iff e' = I(t(e'))^+$ *in* $(\Gamma,\prec)$;
(3) $e = O(s(e))^-$ *in* $(G, \prec_G) \iff e' = O(s(e'))^-$ *in* $(\Gamma,\prec)$;
(4) $e = O(s(e))^+$ *in* $(G, \prec_G) \iff e' = O(s(e'))^+$ *in* $(\Gamma,\prec)$.

The following result shows that for a UPO-graph, there is a CPP-extension uniquely determined by its upward planar order.

Theorem 5.5 *Let $(G, \prec)$ be a UPO–graph. Then, there exists a unique CPP-extension $\phi{:}G \to (\overline{G}, \overline{\prec})$ such that for any $e_1,e_2 \in E(G)$, $e_1 \prec e_2$ implies that $\phi_1(e_1)\overline{\prec}\phi_1(e_2)$.*

Proof We construct a POP-graph $(\overline{G}, \overline{\prec})$ by adding vertices and edges to $(G, \prec)$ just in the ways of Theorem 3.5.

(1) For each $v \in S(G)$, if $U(v) = \emptyset$, we add to G a source v^- and an input edge e_v with $s(e_v) = v^-$, $t(e_v) = v$. Otherwise, we add to G an edge e_v with $s(e_v) = U(v)^-$, $t(e_v) = v$. For both cases, we set the order $e_v = O(v)^- - 1$.

(2) For each $v \in T(G)$, if $D(v) = \emptyset$, we add to G a sink v^+ and an output edge e_v with $s(e_v) = v$, $t(e_v) = v^+$. Otherwise, we add to G an edge e_v with $s(e_v) = v$, $t(e_v) = D(v)^-$. For both cases, we set the order $e_v = I(v)^+ + 1$.

Clearly, the order of adding edges is unimportant and the above construction produces a unique processive graph $\overline{G}$, a unique linear order $\overline{\prec}$ on $E(\overline{G})$ and a unique embedding $\phi : G \to \overline{G}$ which preserves the orders on edges and satisfies (E_1), (E_2), (E_3), (E_4). Iteratively applying Theorem 3.5, we see that $(\overline{G}, \overline{\prec})$ is a UPO–graph.

To show that $(\overline{G}, \overline{\prec})$ is a POP-graph, by Theorem 4.6, it suffices to show that $(\overline{G}, \overline{\prec})$ satisfies (U_4). We prove this by contradiction. Suppose there exists a processive vertex v of $\overline{G}$ with $I(\overline{G}) \cap \overline{O(v)} \neq \emptyset$. Clearly, $v \in V(G)$. Take an edge $e \in I(\overline{G}) \cap \overline{O(v)}$ and set $w = t(e)$. By the construction of $(\overline{G}, \overline{\prec})$, $e \in I(\overline{G})$ implies that $w \in S(G)$, $U(w) = \emptyset$ in $(G, \prec)$, $e = e_w$, and $O(w)^- = e + 1$. Then, $e \in \overline{O(v)}$ implies that $O(v)^-\overline{\prec}O(w)^-\overline{\preceq}O(v)^+$. If $O(v)^+ = O(w)^-$, then $v = w$, and hence $e \in \overline{O(v)} \cap I(v)$ in $(\overline{G}, \overline{\prec})$, which contradicts the fact that $\overline{\prec}$ satisfies (U_2). So we must have $O(v)^-\overline{\prec}O(w)^-\overline{\prec}O(v)^+$, which means $O(w) \cap \overline{O(v)} \neq \emptyset$ in $(\overline{G}, \overline{\prec})$. Then, by (U_3) for $\overline{\prec}$, we have $\overline{O(w)} \subset \overline{O(v)}$ in $(\overline{G}, \overline{\prec})$, which, by the construction of $(\overline{G}, \overline{\prec})$, implies that $\overline{O(w)} \subset \overline{O(v)}$ in $(G, \prec)$, that is, $v \in U(w) \neq \emptyset$ in $(G, \prec)$, a contradiction. Similarly, we can prove that for any processive vertex v of $\overline{G}$, $O(\overline{G}) \cap \overline{I(v)} = \emptyset$.

Now we show the uniqueness of the CPP extension. Suppose $\varphi\colon G \to (G_1, \prec_1)$ is a CPP-extension of G that preserves the upward planar orders. By Lemma 5.3, for any $e \in E(G_1) - \varphi_1(E(G))$, there exists a unique $v \in S(G) \sqcup T(G)$, such that $\{e\} = I(\varphi_0(v))$ or $\{e\} = O(\varphi_0(v))$. Since $(G_1, \prec_1)$ is a UPO–graph, then $e = O(t(e))^- + 1$ or $e = I(s(e))^+ + 1$. Comparing to the construction of $(\overline{G}, \overline{\prec})$, it is not difficult to see that there exists a canonical order-preserving isomorphism $\lambda\colon G_1 \to \overline{G}$ such that $\phi = \lambda \circ \varphi$. Thus, all order-preserving CPP-extensions of $(G, \prec)$ are canonically isomorphic to each other. □

Conversely, a CPP extension always induces an upward planar order.

Proposition 5.6 *Any CPP-extension of an acyclic directed graph G induces an upward planar order on $E(G)$.*

Proof Let $\phi\colon G \to (\Gamma, \prec)$ be a CPP-extension of G. By Proposition 3.4, the induced order $\prec_G$ is a linear order satisfying (U_1) and (U_3). To show that $\prec_G$ satisfies (U_2), by (U_1), it suffices to show that for any processive vertex v of G, $I(v)^+ + 1 = O(v)^-$, which follows from Lemma 5.4. □

As a direct consequence of Theorem 5.5 and Proposition 5.6, the following is our main result in this section.

Theorem 5.7 *For any acyclic directed graph G, there is a bijection between the set of upward planar orders and the set of CPP-extensions.*

6 Justifying UPO-Graph

In this section, we will prove our main result, Theorem 6.1, which shows that upward planar orders indeed characterize upward planarity.

Theorem 6.1 *Any UPO–graph has a unique upward planar drawing up to planar isotopy, and conversely, there is an upward planar order on $E(G)$ for any upward plane graph G.*

The first part follows from Theorems 2.5 and 5.7. For the converse part, we need the following proposition, which is a geometric counterpart of Theorem 5.5.

Proposition 6.2 *Let G be an upward plane graph. Then, there exist a PPG Γ and a (geometric) embedding $\phi\colon G \to \Gamma$, such that*

(1) $\phi_0(V(G)) = V(\Gamma) - (S(\Gamma) \sqcup T(\Gamma))$;
(2) $|E(\Gamma)| = |E(G)| + |S(G)| + |T(G)|$;
(3) *for any* $v \in S(G)$ *and* $w \in T(G)$, $I(\phi_0(v)) \cap O(\phi_0(w)) = \emptyset$;
(4) $e \in E(\Gamma) - (\phi_1(E(G)) \cup I(\Gamma) \cup O(\Gamma))$ *implies that either* $O(s(e))^- < e < O(s(e))^+$ *or* $I(t(e))^- < e < I(t(e))^+$, *where the linear orders are given by the polarization of* Γ.

Proof We want to extend G into a PPG. Let $v_1, \ldots, v_n$ be an ordered list of $V(G)$, with $(X_1, Y_1), \ldots, (X_n, Y_n)$ as their coordinates, such that $Y_1 \geq \ldots \geq Y_n$. Then, G is contained in the box $D = [K-1, L+1] \times [Y_1+1, Y_n-1]$, where $K = min\{X_1, \ldots, X_n\}$, $L = max\{X_1, \ldots, X_n\}$.

Assume $T(G) = \{v_{\alpha_1}, \ldots, v_{\alpha_\mu}\}$ with $1 \leq \alpha_1 < \ldots < \alpha_\mu \leq n$. Clearly, $v_n = v_{\alpha_\mu}$ is a sink. We will inductively eliminate all the sinks of G by adding suitable new edges and vertices.

First, we add a vertex v_n^+ and an edge $h = [v_n, v_n^+]$ to G, where the coordinate of v_n^+ is $(X_n, Y_n - 1)$ and $h = [v_n, v_n^+]$ is the segment with $s(h) = v_n$, $t(h) = v_n^+$. Denote the resulting upward plane graph as G_1.

Then, we move to $v_{\alpha_{\mu-1}}$. If $Y_{\alpha_{\mu-1}} = Y_{\alpha_\mu}$, just as above, we add a vertex $v_{\alpha_{\mu-1}}^+$ and an edge $h_1 = [v_{\alpha_{\mu-1}}, v_{\alpha_{\mu-1}}^+]$ to G_1. Otherwise, $Y_{\alpha_{\mu-1}} > Y_{\alpha_\mu}$. Then, we consider the

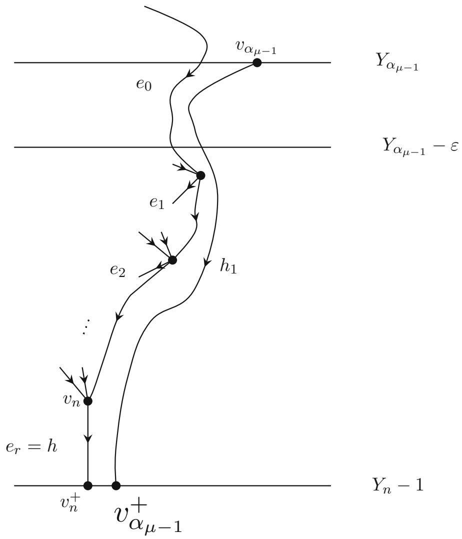


Fig. 11 Drawing of h_1 in subcase (1.1)

horizontal line $y = Y_{\alpha_{\mu-1}}$ and the set of its intersection points with G_1. There are three cases: (1) there is an intersection point on the left of $v_{\alpha_{\mu-1}}$; (2) there is an intersection point on the right of $v_{\alpha_{\mu-1}}$; and (3) $v_{\alpha_{\mu-1}}$ is the unique intersection point of the line with G_1.

Case (1): We consider the strip of the plane delimited by horizontal lines $y = Y_{\alpha_{\mu-1}}$ and $y = Y_{\alpha_{\mu-1}} - \varepsilon$, where $\varepsilon > 0$ is small enough so that the strip contains no vertices in its interior, and the strip is divided by the edges of G_1 into (at least two) connected regions bounded by vertically monotonic curves.

Let e_0 be the edge of G_1 such that it is on the boundary of the region that contains $v_{\alpha_{\mu-1}}$ and on the left of $v_{\alpha_{\mu-1}}$ (the assumption in (1) guarantees the existence of e_0). Then, either $e_0 = h$, or there exists a unique directed path $e_0e_1e_2 \cdots e_r$, such that $e_i = O(s(e_i))^+$ for all $1 \le i \le r$ and $t(e_r) = v_n^+$ (equivalently, $e_r = h$), where the existence and the uniqueness of such path are guaranteed by the ordering of $T(G)$ and the requirement $e_i = O(s(e_i))^+$, respectively. Then, we have two subcases.

Subcase (1.1): For all $0 \le i \le r$, $e_i = I(t(e_i))^+$ with respect to the polarization of G.

In this case, we add to G_1 a vertex $v_{\alpha_{\mu-1}}^+$ with vertical ordinate $Y_n - 1$, on the right of and close enough to v_n^+; add an edge h_1 with $s(h_1) = v_{\alpha_{\mu-1}}$, $t(h_1) = v_{\alpha_{\mu-1}}^+$, and draw it into G_1 as a monotonic curve on the right of and close enough to the directed path $e_0e_1 \cdots e_r$, see Fig. 11.

Subcase (1.2): There exists some $i \in [0, \ldots, r-1]$ such that $e_i < I(t(e_i))^+$, and $e_j = I(t(e_j))^+$ for all $j < i$.

In this case, we add to G_1 an edge h_1 with $s(h_1) = v_{\alpha_{\mu-1}}$, $t(h_1) = t(e_i)$, and draw it into G_1 as a monotonic curve on the right of and close enough to $e_0e_2 \cdots e_i$; see Fig. 12. Clearly, $e_i < h_1 < I(t(e_i))^+$ with respect to the polarization of the resulting upward plane graph.

Case (2): This case is similar to Case (1), and we omit it here.

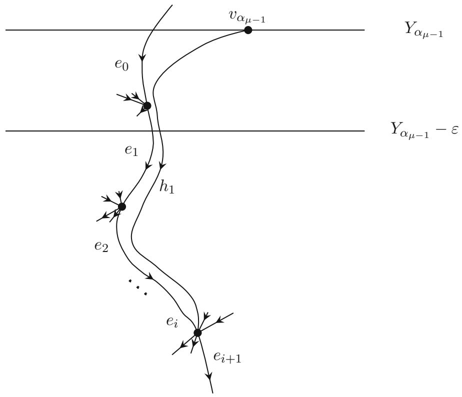


Fig. 12 Drawing of h_1 in case (1.2)

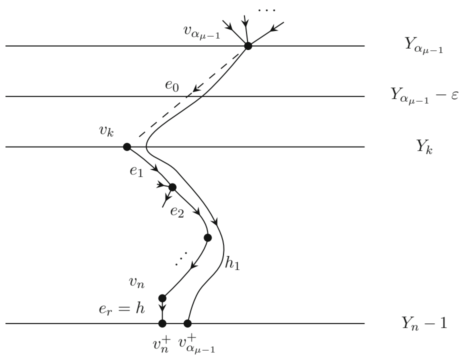


Fig. 13 One possible subcase of case (3)

Case (3): In this case, since $Y_{\alpha_{\mu-1}} > Y_{\alpha_\mu}$, so there must exist a source $v_k \in V(G)$ with $\alpha_{\mu-1} < k < n$ such that there is no edge of G intersecting with the interior of the horizontal strip delimited by horizontal lines $y = Y_{\alpha_{\mu-1}}$ and $y = Y_k$. Take $e_0 = [v_{\alpha_{\mu-1}}, v_k]$ the segment with $s(e_0) = v_{\alpha_{\mu-1}}$ and $t(e_0) = v_k$. Then, we reduce this case to the above case (1) or case (2) of the auxiliary upward plane graph $G_1' = G_1 + \{e_0\}$. Figure 13 shows one possible subcase of case (3), where e_0 is just an auxiliary edge for the drawing of h_1 and is not a really added edge of G_1.

Repeating the above procedure successively for $v_{\alpha_{\mu-2}}, v_{\alpha_{\mu-3}}, \ldots, v_{\alpha_1}$, we can eliminate all the sinks. Similarly, we can eliminate all the sources. As a result, we get a PPG Γ boxed in D and with G as a subgraph. By the construction, the resulting embedding $\psi : G \to \Gamma$ satisfies all the required conditions listed in the proposition. The proof is completed. □

By Theorem 2.5, the (geometric) embedding $\phi : G \to \Gamma$ in Proposition 6.2 induces a CPP-extension of G, which, by Proposition 5.6, implies the converse part of Theorem 6.1.

Remark 6.3 Note that the extension of G in Proposition 6.2 is not unique, so the upward planar order on $E(G)$ is not necessarily unique.

Remark 6.4 Note that if $(G, \prec)$ is an anchored UPO-graph, then for any $v \in S(G)$, $U(v) = \emptyset$, and for any $v \in T(G)$, $D(v) = \emptyset$. By the construction in Theorem 5.5, the CPP extension $\phi : G \to (\overline{G}, \overline{\prec})$ satisfies $E(\overline{G}) - E(G) = I(\overline{G}) \sqcup O(\overline{G})$. Therefore, in the geometric realization of $(G, \prec)$, all sources and sinks of G are drawn on the boundary of the external face.

Using a fundamental result independently due to Fáry [4] and Wagner [12], we may obtain a combinatorial characterization of (non-directed) planar graphs.

Corollary 6.5 *A* (*non-directed*) *graph G has a planar drawing if and only if there exists an orientation on G and an upward planar order on $E(G)$ with respect to the orientation.*

Proof We need only to show the "only if" part, and the other direction is obvious.

We first claim that any simple planar graph Γ has an upward drawing. By Fáry-Wagner theorem, there exists a planar drawing of Γ such that all edges are straight line segment. We may rotate the plane an appropriate angle, so that any horizontal line contains at most one vertex of Γ. This can be done since Γ has only finitely many vertices, and hence only finitely many straight lines will contain more than one vertices. Then, there exists a (unique) orientation of Γ making the resulting drawing an upward planar drawing.

Now the proof follows from the easy fact that a graph G has a planar drawing if and only if the associated simple graph of G has a planar drawing. □

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References

1. Bertolazzi, G., Di Battista, G.: On upward drawings of triconnected digraphs. In: Proceedings of the 7th Annual Symposium on Computational Geometry, pp. 272–280 (1991)
2. Bertolazzi, G., Di Battista, G., Liotta, G., Mannino, C.: Upward drawings of triconnected digraphs. Algorithmica **12**, 476–497 (1994)
3. Di Battista, G., Tamassia, R.: Algorithms for plane representations of acyclic digraphs. Theor. Comput. Sci. **61**(2–3), 175–198 (1988)
4. Fáry, I.: On straight line representation of planar graphs. Acta. Univ. Szeged. Sect. Sci. Math. **11**, 229–233 (1948)
5. Garg, A., Tamassia, R.: Upward planarity testing. Order **12**(2), 109–133 (1995)
6. Hu, S., Lu, X., Ye, Y.: A graphical calculus for semi-groupal categories. Appl. Categ. Struct. arXiv:1604.07276 **(to appear)**
7. Joyal, A., Street, R.: The geometry of tensor calculus. I. Adv. Math. **88**(1), 55–112 (1991)
8. Kelly, D.: Fundamentals of planar ordered sets. Discrete Math. **63**(2), 197–216 (1987)
9. Rival, I.: Reading, drawing, and order. In: Rosenberg, I.G., Sabidussi, G. (eds.) Algebras and Orders, NATO ASI Series (Series C: Mathematical and Physical Sciences), vol. 389, pp. 359–404. Springer, Dordrecht (1993)
10. Trotter, W.T.: Order preserving embeddings of aographs. In: Alavi, Y., Lick, D.R. (eds.) Theory and Applications of Graphs. Lecture Notes in Mathematics, vol. 642, pp. 572–579. Springer, Berlin (1978)
11. Tamassia, R., Tollis, I.G.: A unified approach to visibility representations of planar graphs. Discrete Comput. Geom. **1**(4), 321–341 (1986)
12. Wagner, K.: Bemerkung zum Vierfarbenproblem. Jber. Deutsch. Math. Verein. **46**, 26–32 (1936)